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Square Eigenfunction Approach to the Hamiltonian Structure for a New Hierarchy of Nonlinear Equations

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We deduce the symplectic form for the Hamiltonian structure of a new class of nonlinear equations with the help of square eigenfunctions associated with the corresponding linear problem. The method actually yields two pieces of information simultaneously. One is the structure of the square eigenfunctions, which is of prime importance in the study of the inverse problem. The other is the form of the symplectic structure fixing up the canonical Poisson bracket relation. Finally we discuss some reductions of the initial system and the corresponding change of the Hamiltonian structure and the form of the square eigenfunction.

1. INTRODUCTION

After the pioneering paper of Ablowitz *et al.* (1974) (AKNS), it was felt that one of the most important objects associated with any inverse problem is the square eigenfunction constructed out of the solutions of the Lax equations and the eigenvalue problem satisfied by these eigenfunctions themselves. Incidentally, the construction of square eigenfunctions (SG) is not at all a trivial problem. Except in the simplest AKNS problem, it usually requires much tricky manipulation to obtain the eigenvalue equation of SG. Even in some cases some dependence on the nonlinear field variables or eigenvalue λ are to be assumed. It may be recalled that there exist at present two distinct procedures for arriving at the Hamiltonian structure. One is the method of recurrence relations advocated by the Italian school (Boiti *et al.*, 1982, 1984; Tu, 1982) and the other is that of square eigenfunctions (Newell, 1979). While the former method can lead to a final result with less labor, the latter, though a bit elaborate, gives crucial information about the

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form and equation of square eigenfunctions, which is really a basic ingredient in the process of inverse scattering transform. In this paper we consider a singular (in the λ plane) Lax equation,

$$\psi_x = -i\lambda\sigma_3\psi + \begin{pmatrix} 0 & q_1 \\ q_2 & 0 \end{pmatrix}\psi + \frac{i}{\lambda}\begin{pmatrix} q_3 & q_4 \\ q_5 & q_6 \end{pmatrix}\psi \tag{1}$$

and the hierarchy of equations associated with it. We then deduce the form of square eigenfunction and the equation satisfied by such SGs. If we now couple a suitable time evolution of ψ, then the linear evolution of the scattering data is seen to be mapped on the required Hamiltonian flows. In the course of our derivation we digress to show how one can deduce the form of the square eigenfunctions for the Kaup–Newell problem and derive the corresponding Hamiltonian form, as this problem has not been discussed in this methodology in the literature.

2. FORMULATION

Though our main aim is to study the spectral problem noted above, we first consider the Kaup–Newell (1978) problem, whose Hamiltonian structure was discussed long before by Sasaki (1980) in the recurrence formalism. Here we show first how to obtain the form of the square eigenfunction and deduce the same result. The nontriviality of these results becomes quite evident when one looks into the derivation of Kaup (1984) for the square eigenfunctions in the case of the sine-Gordon equation in laboratory coordinates.

The Kaup–Newell spectral problem is written as

$$\phi_x = i\lambda^2 A\phi + \lambda N\phi \tag{2}$$

where

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}; \qquad N = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}$$

It is now customary to define the Jost functions with the help of the asymptotic conditions

$$\lim_{x\to-\infty}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} e^{i\lambda^2 x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \qquad \lim_{x\to+\infty}\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}; \qquad e^{-i\lambda^2 x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3}$$

With $r = \pm q^*$ and a suitable time evolution of ϕ we can arrive at the derivative nonlinear Schrödinger equation. Let us now start from (2) and write it in component form,

$$\begin{aligned} v_{1x} &= -i\lambda^2 v_1 + \lambda q v_2 \\ v_{2x} &= \lambda r v_1 + i\lambda^2 v_2 \end{aligned} \tag{4}$$

From equations (4) we deduce

$$\begin{aligned}\lambda^2 v_2^2 &= -\frac{i}{2}(v_2^2)_x + i\lambda^2\left(r\int rv_1^2 + r\int qv_2^2\right)\\ \lambda^2 v_1^2 &= \frac{i}{2}(v_1^2)_x - i\lambda^2\left(q\int rv_1^2 + q\int qv_2^2\right)\end{aligned} \tag{4a}$$

under the asymptotically vanishing condition for the potentials. Finally, we get (with $D=\partial/\partial x$)

$$\begin{bmatrix} -D+iq\int rD & -iq\int rD \\ -ir\int rD & D+ir\int qD \end{bmatrix}\begin{pmatrix} v_1^2 \\ v_2^2 \end{pmatrix} = 2i\lambda^2\begin{pmatrix} v_1^2 \\ v_2^2 \end{pmatrix} \tag{5}$$

which is the spectral problem satisfied by the square eigenfunction $(v_1^2, v_2^2)^T$. Since the space part of the Lax pair alone cannot specify the equation under consideration, let us consider the time variation of the scattering data. We follow the usual nomenclature in writing the S-matrix as

$$S=\begin{pmatrix} a(\lambda) & \bar{b}(\lambda) \\ b(\lambda) & -\bar{a}(\lambda) \end{pmatrix}$$

It is then easy to observe that

$$\begin{aligned}\delta a &= \lambda\int_{-\infty}^{+\infty}(\delta q\,\phi_2\psi_2 - \delta r\,\phi_1\psi_1)\,dx\\ \delta b &= \lambda\int_{-\infty}^{+\infty}(-\delta q\,\phi_2\bar{\psi}_2 + \delta r\,\phi_1\bar{\psi}_1)\,dx\\ \delta\bar{b} &= \lambda\int_{+\infty}^{+\infty}(\delta q\,\bar{\phi}_2\psi_2 - \delta r\,\bar{\phi}_1\psi_1)\,dx\\ \delta\bar{a} &= \lambda\int_{-\infty}^{+\infty}(-\delta q\,\bar{\phi}_2\bar{\psi}_2 + \delta r\,\bar{\phi}_1\bar{\psi}_1)\,dx\end{aligned} \tag{6}$$

Let us assume that the time evolution of ϕ is written as

$$\phi_t = Q\phi \tag{7}$$

with

$$Q=\begin{pmatrix} \lambda^3 A(\lambda q) & B(\lambda q) \\ C(\lambda q) & -\lambda^3 A(\lambda q) \end{pmatrix}$$

Following Ablowitz, we can now write the compatibility condition between (7) and (2) as

$$\tilde{S}_x = \lambda\Phi^{-1}N_t\Phi \tag{8}$$

where $Q=\Phi S\Phi^{-1}$. Integration of (8) leads to

$$\tilde{S}=S(-\infty)+\lambda\int_{-\infty}^{+\infty}(\Phi^{-1}N_t\Phi)\,dx \tag{9}$$

Now, from the definition of S we get, for all (x, t),

$$\tilde{S}=\begin{bmatrix} -\lambda^3A\phi_2\phi_1-B\bar{\phi}_2\phi_1 & -\lambda^3A\bar{\phi}_2\bar{\phi}_1-B\bar{\phi}_2^2 \\ +C\bar{\phi}_1\phi_1-\lambda^2A\bar{\phi}_1\phi_2 & +C\bar{\phi}_1^2-\lambda^3A\bar{\phi}_2\bar{\phi}_1 \\ \lambda^3A\phi_2\phi_1+B\phi_2^2 & \lambda^3A\phi_2\bar{\phi}_1+B\phi_2\bar{\phi}_2 \\ -C\phi_1^2+\lambda^3A\phi_1\phi_2 & -C\phi_1\bar{\phi}_1+\lambda^3A\phi_1\bar{\phi}_2 \end{bmatrix} \tag{10}$$

On the other hand, if, for $x\to-\infty$, $B, C\to 0$ and $Z(\lambda, q)\to A(\lambda)$, then

$$\begin{aligned} S(-\infty)&=\lambda^3A(\lambda)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \\ S(+\infty)&=\lambda^3A\begin{pmatrix}a\bar{a}-b\bar{b} & 2\bar{a}\bar{b}\\ 2ab & -(a\bar{a}-b\bar{b})\end{pmatrix} \end{aligned} \tag{11}$$

So if we now write out equation (8) in full, we get

$$\lambda^3A(\lambda)(a\bar{a}+b\bar{b}-1)=\lambda^2\int_{-\infty}^{+\infty}(-q_t\bar{\phi}_2\phi_2+r_t\bar{\phi}_1\phi_1)\,dx \tag{12a}$$

$$\lambda^3A(\lambda)2\bar{a}\bar{b}=\lambda^2\int_{-\infty}^{+\infty}(-q_t\bar{\phi}_2^2+r_t\bar{\phi}_1^2)\,dx \tag{12b}$$

$$\lambda^3A(\lambda)2ab=\lambda^2\int_{-\infty}^{+\infty}(q_t\phi_2^2-r_t\phi_1^2)\,dx \tag{12c}$$

We now consider (12c) and observe from equations (4) and (4a) that

$$\lambda ab=\int_{-\infty}^{+\infty}\left(\frac{i}{2}q_x\phi_2^2-\frac{i}{2}r_x\phi_1^2\right)dx \tag{13}$$

So we immediately arrive at

$$\int_{-\infty}^{+\infty}\left\{(\phi_2^2, \phi_1^2)\begin{pmatrix}q_5\\ -r_t\end{pmatrix}-A(\lambda)(\phi_2^2, \phi_1^2)\begin{pmatrix}q_x\\ -r_x\end{pmatrix}\right\}dx=0$$

or

$$\int_{-\infty}^{+\infty}(\phi_2^2, \phi_1^2)\left[\begin{pmatrix}q_5\\ -r_t\end{pmatrix}-A(\lambda)D\begin{pmatrix}q\\ -r\end{pmatrix}\right]dx=0 \tag{14}$$

which at once implies [under the assumption of analytic character of $A(\lambda)$] that

$$\begin{pmatrix} q_t \\ -r_t \end{pmatrix} + iD \cdot A(\mathscr{L}) \begin{pmatrix} -q \\ r \end{pmatrix} = 0 \tag{15}$$

with

$$\mathscr{L} = \begin{pmatrix} -D + iq \int rD & -iq \int rD \\ -ir \int rD & D + ir \int qD \end{pmatrix}$$

It is interesting to note that if we take $A(\lambda) = \lambda$, (15) reduces to

$$iq_t - q_{xx} + i(q^* q^2)_x = 0 \tag{16}$$

so that (15) yields the symplectic form, if we express $A(\mathscr{L})\binom{-q}{r}$ as the variational derivatives of some conserved polynomials $H_i(q, r)$, which can be considered to be Hamiltonians. In fact, this point has been deduced in Sasaki (1980) and we do not reproduce it here; our main goal was to obtain the symplectic operator

$$\frac{\partial}{\partial x} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

from the considerations of square eigenfunctions.

3. A SPECTRAL PROBLEM WITH A SINGULARITY AT $\lambda = 0$; A NEW HIERARCHY OF EQUATIONS

In this section we consider a new class of nonlinear equations that can be obtained from the spectral problem noted in equation (1). Written in component form, equation (1) reads

$$\begin{aligned} v_{1x} &= -i\lambda v_1 + -\frac{i}{\lambda} q_3 v_1 + \frac{i}{\lambda} q_4 v_2 + q_1 v_2 \\ v_{2x} &= i\lambda v_2 - \frac{i}{\lambda} - q_3 v_2 + -\frac{i}{\lambda} - q_5 v_1 + q_2 v_1 \end{aligned} \tag{17}$$

It is not difficult to see that the following two equations hold due to (17):

$$\begin{aligned} &\frac{i}{2}(v_1^2)_x - iq_1 \int q_2 v_1^2 - iq_1 \int q_1 v_2^2 \\ &+ \frac{i}{\lambda} q_4 \int \left(q_1 v_2^2 + q_2 v_1^2 + \frac{i}{\lambda} q_5 v_1^2 + \frac{i}{\lambda} q_4 v_2^2 \right) \\ &+ \frac{i}{\lambda} q_3 v_1^2 + \frac{i}{\lambda} q_1 \int q_5 v_1^2 + \frac{i}{\lambda} q_1 \int q_4 v_2^2 = \lambda v_1^2 \end{aligned} \tag{18}$$

$$
-\frac{i}{2}(v_2^2)_x + iq_2\int q_1v_2^2 + iq_2\int q_2v_1^2
$$

$$
-\frac{i}{\lambda}q_5\int\left(q_2v_1^2+q_1v_2^2+\frac{i}{\lambda}q_5v_1^2+\frac{i}{\lambda}q_4v_2^2\right)+\frac{i}{\lambda}q_3v_2^2
$$

$$
-\frac{i}{\lambda}q_2\int q_5v_1^2-\frac{i}{\lambda}q_2\int q_4v_2^2=\lambda v_2^2 \tag{19}
$$

from which one can immediaately infer

$$
[\mathscr{L}-\lambda]\Theta=0 \tag{20}
$$

where Θ is the square eigenfunction vector defined as

$$
|\Theta\rangle=\left[\phi_2^2,\ \phi_1^2,\ 2i\lambda^{-1}\int(q_1\phi_2^2+q_2\phi_1^2+i\lambda^{-1}q_4\phi_2^2 + i\lambda^{-1}q_5\phi_1^2),\ i\lambda^{-1}\phi_2^2,\ i\lambda^{-1}\phi_1^2\right]^t \tag{21}
$$

and $\mathscr{L}$ is a 5×5 matrix written as

$$
\mathscr{L}=\begin{bmatrix}
-\frac{1}{2}i\partial_x+iq_2\int q_1 & iq_2\int q_2 & \frac{1}{2}iq_5 & iq_2\int q_4-iq_3 & iq_2\int q_5\\
iq_1\int q_1 & -\frac{1}{2}\partial_x-iq_1\int q_2 & -\frac{1}{2}iq_4 & -iq_1\int q_4 & -iq_1\int q_5-iq_3\\
2i\int q_1 & 2i\int q_2 & 0 & 2i\int q_4 & 2i\int q_5\\
i & 0 & 0 & 0 & 0\\
0 & i & 0 & 0 & 0
\end{bmatrix} \tag{22}
$$

From equation (22) it can be noted that the square eigenfunctions defined depend on the nonlinear field variables and also the spectral parameter, as in Boiti *et al* (1982, 1984) and Tu (1982).

Now, to reproduce a class of nonlinear equations we must specify a time evolution along with (20). Let us put, as in (11),

$$
\psi_t=Q\psi \qquad \text{with} \quad Q=\begin{pmatrix}\lambda A(\lambda) & B(\lambda)\\ C(\lambda) & -\lambda A(\lambda)\end{pmatrix}
$$

If we write equation (1) as

$$
\psi_x=-i\lambda\sigma_3\psi+N\psi+i\lambda^{-1}M\psi
$$

with

$$
N=\begin{pmatrix}0 & q_1\\ q_2 & 0\end{pmatrix},\qquad M=\begin{pmatrix}q_3 & q_4\\ q_5 & -q_3\end{pmatrix} \tag{23}
$$

then proceeding as usual, we deduce

$$S_x = \Phi^{-1}(N_t + i\lambda^{-1}M_t)\Phi \tag{24}$$

where $Q = \Phi S \Phi^{-1}$. So, integrating (24), we get

$$S = S(-\infty) + \int \Phi^{-1}(N_t + i\lambda^{-1}M_t)\Phi\, dx \tag{25}$$

Now, if the fields tend to zero asymptotically, then

$$S(-\infty) = \lambda A\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

$$S(+\infty) = \lambda A\begin{bmatrix}a\bar{a} - b\bar{b} & 2\bar{a}\bar{b}\\ 2ab & -(a\bar{a} - b\bar{b})\end{bmatrix} \tag{26}$$

On the other hand, we have

$$\int \Phi^{-1}(N_t + i\lambda^{-1}M_t)\Phi\, dx = \int \begin{bmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{bmatrix} dx \tag{27}$$

where

$$\begin{aligned}
\alpha_{11} &= -i\lambda^{-1}q_{3t}\bar{\phi}_2\phi_1 - (q_{1t} + i\lambda^{-1}q_{4t})\bar{\phi}_2\phi_1\\
&\quad + (q_{2t} + i\lambda^{-1}q_{5t})\bar{\phi}_2\phi_1 - i\lambda^{-1}q_{3t}\bar{\phi}_1\phi_2\\
\alpha_{12} &= -i\lambda^{-1}q_{3t}\bar{\phi}_2\bar{\phi}_1 - (q_{1t} + i\lambda^{-1}q_{4t})\bar{\phi}_2^2\\
&\quad + (q_{2t} + i\lambda^{-1}q_{5t})\bar{\phi}_1^2 - i\lambda^{-1}q_{3t}\bar{\phi}_1\bar{\phi}_2\\
\alpha_{21} &= i\lambda^{-1}q_{3t}\phi_2\phi_1 + (q_{1t} + i\lambda^{-1}q_{4t})\phi_2^2\\
&\quad - (q_{2t} + i\lambda^{-1}q_{5t})\phi_1^2 + i\lambda^{-1}q_{3t}\phi_1\phi_2\\
\alpha_{22} &= i\lambda^{-1}q_{3t}\phi_2\bar{\phi}_1 + (q_{1t} + i\lambda^{-1}q_{4t})\phi_2\bar{\phi}_2\\
&\quad - (q_{2t} + i\lambda^{-1}q_{5t})\phi_1\bar{\phi}_1 + i\lambda^{-1}q_{3t}\phi_1\bar{\phi}_2
\end{aligned} \tag{28}$$

which, when used in (25), leads to

$$\begin{aligned}
&\lambda A(\lambda)(a\bar{a} - b\bar{b} - 1)\\
&\quad = \int (q_{1t}\bar{\phi}_2\phi_2 + q_{2t}\bar{\phi}_1\phi_1 - i\lambda^{-1}q_{3t}\\
&\qquad \times (\bar{\phi}_2\phi_1 + \bar{\phi}_1\phi_2) + i\lambda^{-1}q_{4t}\bar{\phi}_2\phi_2 + i^{-1}q_{5t}\bar{\phi}_1\phi_1)\, dx
\end{aligned}$$

$$
\begin{aligned}
\lambda A(\lambda)2\bar{a}\bar{b} &= \int (-q_{1t}\bar{\phi}_2^2 + q_{2t}\bar{\phi}_1^2 - 2i\lambda^{-1}q_{3t}\bar{\phi}_2\bar{\phi}_1 \\
&\qquad - i\lambda^{-1}q_{4t}\bar{\phi}_2^2 + i\lambda^{-1}q_{5t}\bar{\phi}_1^2)\,dx \\
\lambda A(\lambda)2ab &= \int (q_{1t}\phi_2^2 - q_{2t}\phi_1^2 + 2i\lambda^{-1}q_{3t}\phi_1\phi_2 \\
&\qquad + i\lambda^{-1}q_{4t}\phi_2^2 - i\lambda^{-1}q_{5t}\phi_1^2)\,dx
\end{aligned}
\tag{29}
$$

Now, from the x part of the Lax pair, we can prove

$$
\begin{aligned}
\lambda ab &= \int \left[iq_5\phi_1^2 + iq_4\phi_2^2 + \frac{i}{2}\,q_2(\phi_1^2)_x + \frac{i}{\lambda}\,q_2q_3\phi_1^2 + \frac{i}{\lambda}\,(q_2q_4 - q_1q_5) \right] \\
&\quad \times \int \left(q_1\phi_2^2 + q_2\phi_1^2 + \frac{i}{\lambda}\,q_5\phi_1^2 + \frac{i}{\lambda}\,q_4\phi_2^2 \right) \\
&\quad - \frac{i}{2}\,q_1(\phi)_{2x}^2 + \frac{i}{\lambda}\,q_1q_3\phi_2^2 \\
&= \int \left[iq_4 + \frac{i}{2}\,q_{1x},\ iq_5 - \frac{i}{2}\,q_{2x},\ -\frac{i}{2}(q_2q_4 - q_1q_5),\ -iq_1q_3,\ -iq_2q_3 \right] |\Theta\rangle \\
&= \langle \eta | \Theta \rangle
\end{aligned}
\tag{30}
$$

Furthermore, we have

$$
|\eta\rangle = \begin{bmatrix} \frac{1}{2}\partial_x & 0 & 0 & i & 0 \\ 0 & -\frac{1}{2}\partial_x & 0 & 0 & i \\ 0 & 0 & 0 & -\frac{1}{2}q_2 & \frac{1}{2}q_1 \\ 0 & 0 & -iq_1 & 0 & 0 \\ 0 & 0 & -iq_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \end{bmatrix} = J|q\rangle \quad \text{(say)} \tag{31}
$$

So, combining equations (30), (31), and the last equation of (29), we get

$$
|q\rangle_t + 2J\Omega(\mathscr{L}^n)|q\rangle = 0 \tag{32}
$$

where $\Omega(\lambda)$ is any analytic function of the spectral parameter λ. Equation (32) actually represents the hierarchy of nonlinear equations generated by our spectral problem.

4. VARIATIONAL EQUATIONS AND HAMILTONIAN STRUCTURE

In this section we want to prove that the quantities $\mathscr{L}^n|q\rangle$ are nothing but variational derivatives of some conserved quantities. From the basic definition of the scattering matrix and the formulas for the variation of the parameter, we deduce

$$\begin{aligned}\delta \log a &= \frac{\delta a(\lambda)}{a} \\ &= \int \Big[i\lambda^{-1}\delta q_3 \left(\frac{\phi_1\psi_2}{a} + \frac{\phi_2\psi_1}{a} \right) \\ &\quad + (\delta q_1 + i\lambda^{-1}\delta q_2)\left(\frac{\phi_2\psi_2}{a} \right) - (\delta q_2 + i\lambda^{-1}\delta q_5)\left(\frac{\phi_1\psi_1}{a} \right) \Big]\, dx \end{aligned} \tag{33}$$

If we use

$$\phi_1\psi_2 = \int dx\, \left(q_2\phi_1\psi_1 + q_1\phi_2\psi_2 + \frac{i}{\lambda} q_5\phi_1\psi_1 + \frac{i}{\lambda} q_4\phi_2\psi_2 \right) \tag{34}$$

then we get

$$\begin{aligned}\frac{\delta a}{a} &= \int \lambda \Big[2i\lambda^{-1}\delta q_3 \int \Big(q_2 \frac{\phi_1\psi_1}{\lambda a} + q_1 \frac{\phi_2\psi_2}{\lambda a} \\ &\quad + -\frac{i}{\lambda} q_5 \frac{\phi_1\psi_1}{\lambda a} + \frac{iq_4}{\lambda} \frac{\phi_2\psi_2}{a} \Big) \\ &\quad + (\delta q_1 + i\lambda^{-1}\delta q_2)\left(\frac{\phi_2\psi_2}{a} \right) \\ &\quad - (\delta q_2 + i\lambda^{-1}\delta q_5)\left(\frac{\phi_1\psi_1}{a} \right) \Big] \end{aligned} \tag{35}$$

Set

$$\begin{aligned}|K\rangle = \Big[&\frac{\phi_2\psi_2}{a}, -\frac{\phi_1\psi_1}{a}, 2i\lambda^{-1} \int \Big(q_2 \frac{\phi_1\psi_1}{a} + q_1 \frac{\phi_2\psi_2}{a} \\ &+ \frac{i}{\lambda} q_5 \frac{\phi_1\psi_1}{\lambda a} + \frac{i}{\lambda} q_4 \frac{\phi_2\psi_2}{a} \Big), i\lambda^{-1} \frac{\phi_2\psi_2}{\lambda a}, i\lambda^{-1} \frac{\phi_1\psi_1}{a} \Big] \end{aligned} \tag{36}$$

so that from (35) we can infer

$$\text{grad } q_1(\ln a) = \lambda |K\rangle \tag{37}$$

where

$$\text{grad } q_i = \left[\frac{\delta}{\delta q_1}, \frac{\delta}{\delta q_2}, \frac{\delta}{\delta q_3}, \frac{\delta}{\delta q_4}, \frac{\delta}{\delta q_5} \right]^T$$

Now, proceeding as in the case of square eigenfunctions, we have

$$
\begin{aligned}
&L|K\rangle = \lambda|K\rangle + \gamma F(q) + \gamma'(\lambda)\bar{F}(q) \\
&F(q) = (q, q, 0, 0, 0, 0)^T \\
&\bar{F}(q) = (0, 0, 0, q_4, q_5)^T
\end{aligned}
\tag{38}
$$

$$
\gamma = \begin{pmatrix} 0 & i & 0 & 0 & 0 \\ -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}; \qquad \gamma'\lambda = \begin{pmatrix} 0 & 0 & 0 & 0 & -\lambda^{-1} \\ 0 & 0 & 0 & \lambda^{-1} & 0 \\ 0 & 0 & i\lambda^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{39}
$$

Now under the assumption of analyticity we set

$$
K = \sum_{n=1}^{\infty} \lambda^{-n} K_n
$$

which, when plugged into (38), yields

$$
K_n = L^{n-1}K_1 - L^{n-2}F'(q) \tag{40}
$$

Now we define the Hamiltonian as

$$
H = -i\lambda + i\lambda^{-1}q_3 + (q_1 + i\lambda^{-1}q_4)v_2/v_1 \tag{41}
$$

But from the Lax equation we know that $y = v_2/v_1$ satisfies the Riccati equation

$$
y_x = 2i\lambda y - \frac{2i}{\lambda} q_3 y + \frac{i}{\lambda} q_5 + q_2 - \frac{i}{\lambda} q_4 y^2 - q_1 y^2 \tag{42}
$$

Expanding

$$
y = \sum_{0}^{\infty} \lambda^{-n} y_n; \qquad H = \sum_{n=1} \lambda^{-n} H_n; \qquad a = \sum_{0}^{\infty} \lambda^{-n} I_n \tag{43}
$$

we get from (37), (41), and (42)

$$
\operatorname{grad} q_i \cdot I_n = L^n K_1 - L^{n-1}F'(q) \tag{44}
$$

Finally we arrive at

$$
q_t = J \frac{\delta}{\delta u} H \tag{45}
$$

giving the Hamiltonian structure of the said hierarchy of equations with J as the simplectic structure and $|\Theta\rangle$ as the basic square eigenfunction.

5. REDUCTION OF THE SYSTEM

In many physical situations it often occurs that the dependent set of variables are not all independent, but are constrained by some algebraic relations and so the actual number of nonlinear variables is reduced. It also happens that the nonlinearity increases due to such reductions. Here we show how such a constraint affects the structure of the square eigenfunction, their operator, and hence the Hamiltonian structure.

Let us again consider the spectral problem (1) and introduce the condition

$$q_3^2+q_4q_5=\gamma^2 \tag{46}$$

where γ is constant both in space and time. The constraint equation (46) leads to

$$q_{3t}=-\frac{q_5}{2q_3}\,q_{4t}-\frac{q_4}{2q_3}\,q_{5t}$$

Substituting in the last equation of (29), we get

$$\lambda A(\lambda)2ab=\int\Bigg[q_{1t}\phi_2^2-q_{2t}\phi_1^2+2i\lambda^{-1}\left(-\frac{q_5}{2q_3}\,q_{4t}-\frac{q_4}{2q_3}\,q_{5t}\right)\phi_1\phi_2$$
$$+i\lambda^{-1}q_{4t}\phi_2^2-i\lambda^{-1}q_{5t}\phi_1^2\Bigg]\,dx \tag{47}$$

which can be written as

$$=\int\langle\sigma|q_t\rangle_R\,dx \tag{48}$$

where $\langle\sigma|$ is the square eigenfunction of the reduced set defined by

$$\langle\sigma|=(\phi_2^2,\,\phi_1^2,\,\chi_1,\,\chi_2) \tag{49}$$

$$\chi_1=i\lambda^{-1}\phi_2^2-i\lambda^{-1}\frac{q_5}{q_3}\int\left(q_1\phi_2^2+q_2\phi_1^2+\frac{i}{\lambda}\,q_5\phi_1^2+\frac{i}{\lambda}\,q_4\phi_2^2\right)$$

$$\chi_2=i\lambda^{-1}\phi_1^2+i\lambda^{-1}\frac{q_4}{q_3}\int\left(q_1\phi_2^2+q_2\phi_1^2+\frac{i}{\lambda}\,q_5\phi_1^2+\frac{i}{\lambda}\,q_4\phi_2^2\right)$$

with

$$|q\rangle_R=\begin{bmatrix}q_{1t}\\-q_{2t}\\q_{4t}\\-q_{5t}\end{bmatrix}$$

Using this condition in equations (18) and (19), we obtain

$$\mathscr{L}=\begin{bmatrix} -\frac{i}{2}D+iq_2Iq_1 & iq_2Iq_2 & -iq_3+iq_2Iq_4 & iq_2Iq_5 \\ -iq_1Iq_1 & \frac{i}{2}D-iq_1Iq_1 & -iq_1Iq_4 & -iq_3-iq_1Iq_t \\ i-i\frac{q_5}{q_3}Iq_1 & -i\frac{q_5}{q_3}Iq_2 & -i\frac{q_5}{q_3}Iq_4 & -i\frac{q_5}{q_3}Iq_5 \\ i\frac{q_4}{q_3}Iq_1 & i+i\frac{q_4}{q_3}Iq_2 & i\frac{q_4}{q_3}Iq_4 & i\frac{q_4}{q_3}Iq_5 \end{bmatrix} \tag{50}$$

along with

$$(\mathscr{L}-\lambda)|\sigma\rangle=0 \tag{50}$$

To obtain the time evolution, we can again evaluate λab from the asymptotic values of ϕ, $\bar{\phi}$, ψ, $\bar{\psi}$, which yields

$$\begin{aligned}\lambda ab &= \int (\phi_1\phi_2)_x \\ &= \int \Big[iq_5\phi_1^2+iq_4\phi_2^2+\frac{i}{2}q_2(\phi_1^2)_x \\ &\quad +\frac{i}{\lambda}q_2q_3\phi_1^2-\frac{i}{2}q_1(\phi_1^2)_x+\frac{i}{\lambda}q_2q_3\phi_2^2 \\ &\quad +\frac{i}{\lambda}(q_2q_4-q_1q_5)\int\Big(q_1\phi_2^2+q_2\phi_1^2+\frac{i}{\lambda}q_3\phi_1^2+-\frac{i}{\lambda}q_4\phi_2^2\Big)\Big] \\ &= \int \langle\eta|\sigma\rangle\,dx\end{aligned}$$

with

$$\langle\eta|=(iq_4+\tfrac{1}{2}iq_{1x},\ iq_5-\tfrac{1}{2}iq_{+x},\ -iq_2q_3,\ -iq_2q_3)$$

and since we can write

$$|\eta\rangle=N|q\rangle_R$$

where

$$N=\begin{pmatrix} \frac{1}{2}D & 0 & i & 0 \\ 0 & -\frac{1}{2}D & 0 & i \\ -iq_3 & 0 & 0 & 0 \\ 0 & -iq_3 & 0 & 0 \end{pmatrix}, \qquad |q\rangle_R=\begin{pmatrix} q_1 \\ q_2 \\ q_4 \\ q_5 \end{pmatrix}$$

the hierarchy of equations is given by

$$|q_R\rangle_t+2N\Omega(\mathscr{L}^n)|q_R\rangle=0$$

Proceeding as before, one can again prove that $\mathscr{L}^n|q\rangle_R$ are variational derivatives of some Hamiltonian and so the reduced hierarchy is a set of nonlinear equations that is Hamiltonian with the symplectic structure determined by N. At this point it will be interesting to note that the whole set of equations is changed if the time evolution is prescribed in a different manner. If the time part is governed by the i/λ term, then the new set of equations is given by

$$|q_R\rangle_t + 2J''\Omega(\mathscr{L}^n)|q\rangle_R = 0$$

with

$$J'' = \begin{pmatrix} 0 & 0 & i/q_3 & 0 \\ 0 & 0 & 0 & i/q_3 \\ -i & 0 & -\frac{1}{2}iDi/q_3 & 0 \\ 0 & 0 & 0 & \frac{1}{2}Di/q_3 \end{pmatrix}$$

so that in each case one can generate the symplectic form J'', N, etc., very easily from the construction of the square eigenfunctions and also it is not difficult to connect it to the variational derivatives of some conserved quantities with respect to the fields. As an example of equations generated, we observe that for $n = 1$ in equation (32) we get

$$iq_{1t} = q_{1x} + 2q_4$$
$$iq_{2t} = q_{2x} - 2q_5$$
$$iq_{3t} = q_1 q_5 - q_2 q_4$$
$$iq_{4t} = -2q_1 q_3$$
$$iq_{5t} = 2q_2 q_3$$

which is the same set of equations as obtained by Boiti *et al.* (1982, 1984).

6. CONCLUSION

We have shown how to construct the square eigenfunctions in the case of a Lax operator other than that of the simplest AKNS form. These eigenfunctions play a pivotal role in the main theme of the inverse scattering transform. Our approach also serves to obtain the symplectic structure in relation to the Hamiltonian form. We have also deduced the symplectic form in the case of the Kaup–Newell problem as a prototype example. In all other cases these eigenfunctions have a complicated dependence on the nonlinear field variables and the spectral parameter. We have reproduced the same set of equations as in Boiti *et al.* (1982, 1984) and Tu (1982), but our approach has extra information regarding the structure of the square eigenfunctions. A few years back it was shown by Fokes *et al.* (1982) that this square eigenfunction can even be used to obtain the hereditary operator for the *LB* symmetries. Work in this direction is in progress and will be reported elsewhere.

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